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Delayed feedback on the 3-D chaotic system only with two stable node-foci

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we investigate the effect of delayed feedbacks on the 3-D chaotic system only with two stable node-foci by Yang et al. The stability of equilibria and the existence of Hopf bifurcations are considered. The explicit formulas determining the direction, stability and period of the bifurcating periodic solutions are obtained by employing the normal form theory and the center manifold theorem. Numerical simulations and experimental results are given to verify the theoretical analysis. Hopf bifurcation analysis can explain and predict the periodic orbit in the chaotic system with direct time delay feedback. We also find that the control law can be applied to the chaotic system only with two stable node-foci for the purpose of control and anti-control of chaos. Finally, some numerical simulations are given to illustrate the effectiveness of the results found.

1. Introduction

In 1963, Lorenz found the first chaotic attractor in a simple three dimensional autonomous system [1]. For a generic three-dimensional smooth quadratic autonomous system, Sprott [2–4] found by exhaustive computer searching 19 simple chaotic systems with none, one equilibrium or two equilibria. It is very important to note that some classical 3-D autonomous chaotic systems have three particular fixed points: one saddle and two unstable saddle-foci (for example, Lorenz system [1], Chen system [5], Lü system [6], the conjugate Lorenz-type system [7]). The other 3-D chaotic systems, such as diffusionless Lorenz equations [8] and Burke–Show system [9], have two unstable saddle-foci. In 2008, Yang and Chen found another 3-D chaotic system with three fixed points: one saddle and two stable equilibria [10]. Many theoretical analyses and numerical simulations about these systems are shown in [11–21].

Without unstable equilibria, the Šilnikov condition is violated, and it is not of great significance either because that condition is known to be sufficient but certainly not necessary for chaos. In 2010, Yang et al. [22] introduced and analyzed a new 3-D chaotic system with six terms including only two quadratic terms in a form very similar to the Lorenz, Chen, Lü and Yang–Chen systems, but it has only two fixed points: two stable node-foci. Some questions about periodic, homoclinic and heteroclinic orbits and classification of chaos, are related to the dynamics of some dynamical systems. The type of chaotic systems is investigated further analytically and numerically in [23,24].

Therefore, understanding the local and the global characteristics of the chaotic dynamical systems is of great importance, because this effort often gives hints for generating/eliminating chaos and indicates the potential applications. Recently the trend of analyzing and understanding chaos has been extended to controlling and utilizing chaos. The main goal of chaos control was to eliminate chaotic behavior and to stabilize the chaotic system at one of the system's equilibrium points. More specially, when it is useful, we want to generate chaos intentionally. Until now, many advanced theories and methodologies have been developed for controlling chaos. Many scientists have more concerns with delayed control [25,26]. The existing

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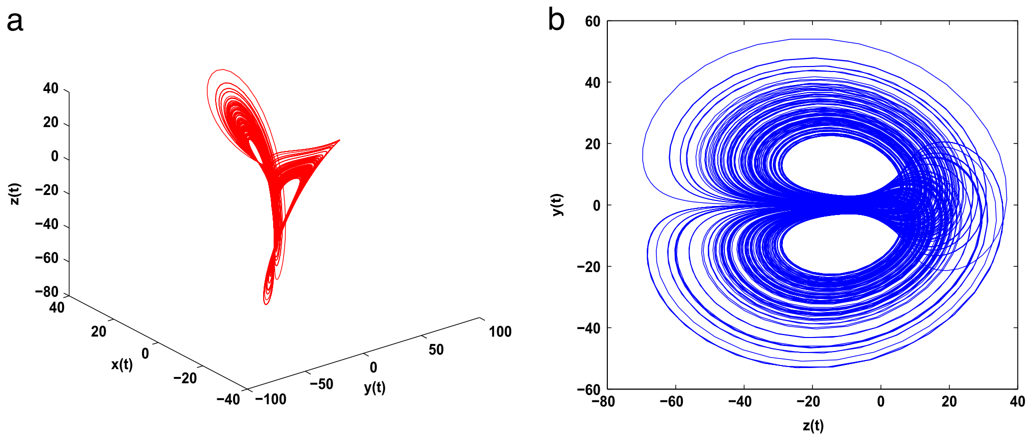

Fig. 1. Parameter values $(a, b, c) = (10, 100, 10.3)$ and initial values $(0.98, -1.82, -0.49)$: (a) Chaotic attractor of system (1) when the equilibria $E_{1,2}$ are both asymptotically stable; (b) Projection of (a) into $y - z$ plane.

control method can be classified, mainly, into two categories. The first one, the OGY method developed by Ott et al. [27] in the 1990s has completely changed the chaos research topic. The second one, proposed by Pyragas [28,29], used time-delayed controlling forces. Compared with the first one, it is much simpler and more convenient in controlling chaos in a continuous dynamics system. Here, we mainly study the system only with two stable node-foci proposed by Yang et al. [22] with delay. Yang et al. described this uncontrolled system by the following three-dimensional smooth autonomous system [22].

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = -cy - xz \\ \dot{z} = -b + xy, \end{cases} \tag{1}$$

where a, b and c are positive real parameters. It has two equilibria

$$E_1 : (x_0, y_0, z_0) = \left(\sqrt{b}, \sqrt{b}, -c\right) \quad \text{and} \quad E_2 : (-x_0, -y_0, z_0) = \left(-\sqrt{b}, -\sqrt{b}, -c\right).$$

In particular, for parameter values $(a, b, c) = (10, 100, 10.3)$, three characteristic values of the Jacobian of the linearized equation evaluated at the equilibria point $E_{1,2}$ are: $\lambda_1 = -20.2411$, $\lambda_{2,3} = -0.0294 - 9.9402i$. The chaotic attractor and its projection in the $y - z$ plane are shown in Fig. 1(a) and (b), respectively. Therefore, system (1) has a chaotic attractor coexisting with two stable node-foci.

The purpose of the present paper is to investigate system (1) with direct time delay feedback (DTDF) analytically and numerically. Our analytical results show that the stability changes as the delays vary. Meanwhile, when all the equilibria are asymptotically stable, the chaotic attractor is converted into a stable steady state, an unstable periodic orbit or another chaotic attractor again when the delay passes through some values.

This paper is organized as follows. In Section 2, a model of system (1) with DTDF is created. The stability and the existence of Hopf bifurcation parameter are determined. In Section 3, based on the normal form method and the center manifold theorem, the direction, stability and the period of the bifurcating periodic solutions are analyzed. To verify the theoretic analysis, numerical simulations are given in Section 4. Finally, Section 5 concludes with some discussions.

2. System (1) with DTDF and existence of Hopf bifurcation

In this section, the controlled system by time-delayed controlling forces proposed by Pyragas [10,11] is designed as follows:

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = -cy - xz + k[y(t - \tau) - y] \\ \dot{z} = -b + xy, \end{cases} \tag{2}$$

where τ is the time delay, k is the gain of the time delay feedback.

Due to the symmetry of E_1 and E_1, it is sufficient to analyze the stability of $E_1(x_0, y_0, z_0)$. By the linear transform

$$\begin{cases} x_1 = x - x_0, \\ y_1 = x - y_0, \\ z_1 = x - z_0, \end{cases}$$

the linear equation of the controlled system (2) is

$$\begin{cases}\dot{x}_1 = a(y_1 - x_1),\\ \dot{y}_1 = cx_1 - cy_1 - \sqrt{b}z_1 + k[y_1(t-\tau) - y_1] - x_1z_1,\\ \dot{z}_1 = \sqrt{b}x_1 + \sqrt{b}y_1 + x_1y_1.\end{cases} \tag{3}$$

The associated characteristic equation of the linearized system is

$$\lambda^3 + (a+c+k)\lambda^2 + (b+ak)\lambda + 2ab - (\lambda^2 + a\lambda)ke^{-\lambda\tau} = 0. \tag{4}$$

When $\tau = 0$, Eq. (4) becomes

$$\lambda^3 + (a+c)\lambda^2 + b\lambda + 2ab = 0. \tag{5}$$

According to the Routh–Hurwitz criterion, Eq. (5) has three roots with negative real parts under the following condition:

$$b > 0, \qquad c > a. \tag{6}$$

Therefore, the two equilibria E_1 and E_2 are both local stable nodes or node-foci.

Now we can rewrite the above Eq. (4) as

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + (b_2\lambda^2 + b_1\lambda)e^{-\lambda\tau} = 0, \tag{7}$$

where $a_2 = a+c+k$, $a_1 = b+ak$, $a_0 = 2ab$, $b_2 = -k$, $b_1 = -ak$.

According to the Hopf bifurcation theory, we know that the system (2) will undergo a Hopf bifurcation if the corresponding characteristic equation has a pair of purely imaginary roots. Suppose that the imaginary root is $i\omega$. Substituting $i\omega$ into Eq. (7), we have

$$-\omega^3 i - a_2\omega^2 + a_1\omega i + a_0 + (-b_2\omega^2 + b_1\omega i)[\cos(\omega\tau) - i\sin(\omega\tau)] = 0. \tag{8}$$

Separating the real and the imaginary parts, we have

$$\begin{cases}\omega^3 - a_1\omega = b_2\omega^2\sin(\omega\tau) + b_1\omega\cos(\omega\tau),\\ a_2\omega^2 - a_0 = b_1\omega\sin(\omega\tau) - b_2\omega^2\cos(\omega\tau),\end{cases} \tag{9}$$

which lead to

$$\omega^6 + (a_2^2 - b_2^2 - 2a_1)\omega^4 + (a_1^2 - 2a_0a_2 - b_1^2)\omega^2 + a_0^2 = 0. \tag{10}$$

Let $z = \omega^2$ and denote $p = a_2^2 - b_2^2 - 2a_1$, $q = a_1^2 - 2a_0a_2 - b_1^2$, and $r = a_0^2$, then Eq. (10) becomes

$$z^3 + pz^2 + qz + r = 0. \tag{11}$$

Let

$$h(z) = z^3 + pz^2 + qz + r.$$

From Eq. (11), we have

$$h'(z) = 3z^2 + 2pz + q.$$

Denote $\Delta = p^2 - 3q$. When $r > 0$ and $\Delta > 0$, the equation

$$3z^2 + 2pz + q = 0$$

has two real roots

$$z_1^* = \frac{-p + \sqrt{\Delta}}{3} \quad \text{and} \quad z_1^* = \frac{-p - \sqrt{\Delta}}{3}.$$

Noticing that $\lim_{z\to+\infty} h(z) = +\infty$ and $r = a_0^2 > 0$, we introduce the following results which was proved by [30].

Lemma 2.1. *For the polynomial equation (11), we have the following results:*

(1) *if $\Delta = p^2 - 3q \le 0$, then Eq. (11) does not have positive real roots;*

(2) *if $\Delta = p^2 - 3q > 0$, then Eq. (11) has positive roots if and only if $z_1^* = \frac{-p+\sqrt{\Delta}}{3} > 0$ and $h(z_1^*) \le 0$.*

Without loss of generality, we give the following assumption:

$$\varDelta = p^2 - 3q > 0, \qquad z_1^* = \frac{-p + \sqrt{\varDelta}}{3} > 0, \quad h(z_1^*) < 0. \tag{12}$$

If (12) holds, then Eq. (11) has two positive roots z_1 and z_2. Suppose $z_1 < z_2$, then $h'(z_1) < 0$, $h'(z_2) > 0$. Substituting $\omega_k = \sqrt{z_k}$ $(k = 1, 2)$ into Eq. (9), we have

$$\tau_k^j = \begin{cases} \dfrac{1}{\omega_k}\left[\arccos(P) + 2j\pi\right], & Q \geq 0, \\ \dfrac{1}{\omega_k}\left[2\pi - \arccos(P) + 2j\pi\right], & Q < 0, \end{cases} \tag{13}$$

where

$$P = \frac{b_1\omega_k^2 - a_1b_1 + a_0b_2 - a_2b_2\omega_k^2}{b_2^2\omega_k^2 + b_1^2}, \qquad Q = \frac{b_2\omega_k^4 - a_1b_2\omega_k^2 + a_2b_1\omega_k^2 - a_0b_1}{\omega_k(b_2^2\omega_k^2 + b_1^2)},$$

and $j = 0, 1, \ldots$.

Lemma 2.2. *If (12) holds, when $\tau = \tau_k^j$ $(k = 1, 2;\ j = 0, 1, 2, \ldots)$, then (7) has a pair of pure imaginary roots $i\omega_k$, and all the other roots of (7) have nonzero real parts.*

Lemma 2.3. *If (12) holds, then we have the following transversality conditions: $\left[\frac{d(\mathrm{Re}\lambda)}{d\tau}\right]_{\tau=\tau_1^j}^{-1} < 0$, $\left[\frac{d(\mathrm{Re}\lambda)}{d\tau}\right]_{\tau=\tau_2^j}^{-1} > 0$, where $j = 0, 1, \ldots$.*

Proof. Substituting $\lambda(\tau)$ into Eq. (7) and taking the derivative with respect to τ, we obtain

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{3\lambda^2 + 2a_2\lambda + a_1}{\lambda(b_2\lambda^2 + b_1\lambda)e^{-\lambda\tau}} + \frac{2b_2\lambda + b_1}{\lambda(b_2\lambda^2 + b_1\lambda)} - \frac{\tau}{\lambda}.$$

From (9), we have

$$\begin{aligned}
\left[\frac{d(\mathrm{Re}\lambda)}{d\tau}\right]_{\tau=\tau_k^j}^{-1} &= \mathrm{Re}\left[\frac{3\lambda^2 + 2a_2\lambda + a_1}{\lambda(b_2\lambda^2 + b_1\lambda)e^{-\lambda\tau}}\right]_{\tau=\tau_k^j} + \mathrm{Re}\left[\frac{2b_2\lambda + b_1}{\lambda(b_2\lambda^2 + b_1\lambda)}\right]_{\tau=\tau_k^j} \\
&= \mathrm{Re}\left[\frac{3\lambda^2 + 2a_2\lambda + a_1}{-\omega_k^2(b_1 + ib_2\omega_k)}(\cos(\omega_k\tau_k^j) + i\sin(\omega_k\tau_k^j))\right] + \mathrm{Re}\left[\frac{2b_2\lambda + b_1}{-\omega_k^2(b_1 + ib_2\omega_k)}\right] \\
&= \frac{z_k}{\varLambda}[3\omega_k^4 + 2(a_2^2 - b_2^2 - 2a_1)\omega_k^2 + a_1^2 - 2a_0a_2 - b_1^2] \\
&= \frac{z_k}{\varLambda}h'(z_k),
\end{aligned} \tag{14}$$

where $\varLambda = \omega_k^4(b_1^2 + b_2^2\omega_k^2)$. Since $z_k > 0$, we conclude that $\left[\frac{d(\mathrm{Re}\lambda)}{d\tau}\right]_{\tau=\tau_k^j}^{-1}$ and $h'(z_k)$ have the same sign. Note that $h'(z_1) < 0$, and $h'(z_2) > 0$, then the proof is complete. □

Now we apply the Hopf bifurcation theorem for functional differential equations [31] and can get the following results.

Theorem 2.4. *Suppose that (6) and (12) is satisfied, then system (2) under goes a Hopf bifurcation at the equilibria $E_{1,2}$ when $\tau = \tau_k^j$ $(k = 1, 2; j = 0, 1, 2, \ldots)$. Moreover, if $\tau_1^0 > \tau_2^0$, then there exists $m \in N$ such that $\tau_2^0 < \tau_1^0 < \tau_2^1 < \tau_1^1 < \cdots < \tau_2^m < \tau_1^m < \tau_2^{m+1} < \tau_2^{m+2} < \tau_1^{m+1}$, and equilibria $E_{1,2}$ of system (2) is asymptotically stable for $\tau \in [0, \tau_2^0)\bigcup(\tau_1^0, \tau_2^1)\bigcup\cdots\bigcup(\tau_1^{m-1}, \tau_2^m)\bigcup(\tau_1^m, \tau_2^{m+1})$ and unstable for $\tau \in [\tau_2^0, \tau_1^0)\bigcup(\tau_2^1, \tau_1^1)\bigcup\cdots\bigcup(\tau_2^m, \tau_1^m)\bigcup(\tau_2^{m+1}, +\infty)$. Furthermore, system (2) undergoes a Hopf bifurcation at the equilibria $E_{1,2}$ when $\tau = \tau_k^j$ $(k = 1, 2; j = 0, 1, \ldots)$.*

Remark. Theorem 2.4 can be applied to system (2) for the purpose of control and anti-control of chaos. When the delay passes through certain critical values, the chaotic attractor only with two stable node-foci may be converted into a stable steady state, an unstable periodic orbit or another chaotic attractor again.

3. Direction and stability of Hopf bifurcation

In the previous section, we obtained the conditions under which the Hopf bifurcation occurs. In this section, the bifurcation direction and the stability of the bifurcations are analyzed using the central manifold theorem. We assume that system (2) always undergoes Hopf bifurcation at the equilibrium E_+ for $\tau = \tau_k$.

Let $u_1 = x - x_0,\ u_2 = y - y_0,\ u_3 = z - z_0,\ \bar{u}_i(t) = u_i\,(\tau t),\ \tau = \mu + \tau_k$, and dropping the bars for simplification of notations. The nonlinear system (2) can be transformed into an FDE in $C \in C([-1, 0], R^3)$ as

$$\dot{u}(t) = L_\mu(u_t) + f(\mu, u_t), \tag{15}$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^T \in R^3$, and $L_\mu : C \to R^3, f : R \times C \to R$ are given, respectively, by

$$L_\mu(\phi) = (\mu + \tau_k)\begin{pmatrix} -a & a & 0 \\ c & -k-c & -\sqrt{b} \\ \sqrt{b} & \sqrt{b} & 0 \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} + (\mu + \tau_k)\begin{pmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix},$$

and

$$f(\mu, u_t) = (\mu + \tau_k)\begin{pmatrix} 0 \\ -\phi_1(0)\phi_3(0) \\ \phi_1(0)\phi_2(0) \end{pmatrix}.$$

Based on the Riesz representation theorem, there is a bounded variation function $\eta(\theta, \mu)$ in $\theta \in [-1, 0]$ such that

$$L_\mu(\phi) = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), \quad \text{for } \phi \in C.$$

In fact, we can choose

$$\eta(\theta, \mu) = (\mu + \tau_k)\begin{pmatrix} -a & a & 0 \\ c & -k-c & -\sqrt{b} \\ \sqrt{b} & \sqrt{b} & 0 \end{pmatrix}\delta(\theta) - (\mu + \tau_k)\begin{pmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{pmatrix}\delta(\theta + 1),$$

where $\delta(\cdot)$ is a Dirac function.

For $\phi \in C([-1, 0], R^3)$, define

$$A(\mu)\,\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \displaystyle\int_{-1}^{0} d\eta(\theta, s)\phi(s), & \theta = 0, \end{cases}$$

and

$$R(\mu)\,\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\mu, \phi), & \theta = 0. \end{cases}$$

For convenience, we can write system (15) into an operate equation

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t, \tag{16}$$

where $u_t(\theta) = u(t + \theta), \theta \in [-1, 0]$. For $\psi \in C^1([0, 1], (R^3)^*)$, define

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0, 1], \\ \displaystyle\int_{-1}^{0} d\eta^T(t, 0)\psi(-t), & s = 0, \end{cases}$$

and a bilinear inner product

$$\langle\psi, \phi\rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi, \tag{17}$$

where $\eta(\theta) = \eta(\theta, 0)$. Obviously A^* and $A(0)$ are adjoint operators. By the discussion in Section 2, we know that $\pm i\omega_k\tau_k$ are eigenvalues of $A(0)$. Thus they are eigenvalues of A^*. We need to calculate the eigenvectors of $A(0)$ and A^* corresponding to $i\omega_k\tau_k$ and $-i\omega_k\tau_k$, respectively. Let $q(\theta) = (1, \alpha, \beta)^T e^{i\theta\omega_k\tau_k}$ is the eigenvectors of $A(0)$. i.e. $A(0)q(\theta) = i\omega_k\tau_k q(\theta)$, then we have

$$\begin{pmatrix} i\omega_k + a & -a & 0 \\ -c & i\omega_k + k + c - ke^{-i\omega_k\tau_k} & \sqrt{b} \\ -\sqrt{b} & -\sqrt{b} & i\omega_k \end{pmatrix} q(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It is easy to obtain

$$q(0) = (1, \alpha, \beta)^T = \left(1, \frac{a + i\omega_k}{a}, \frac{\sqrt{b}(2a + i\omega_k)}{ia\omega_k}\right)^T.$$

Similarly, we can suppose that $q^*(s) = D(1, \alpha^*, \beta^*)e^{is\omega_k\tau_k}$ is the eigenvector of A^* corresponding to $-i\omega_k\tau_k$. From the definition of A^*, we have

$$q^*(s) = D(1, \alpha^*, \beta^*)e^{is\omega_k\tau_k} = D\left(1, \frac{(\omega_k^2 + ia)\omega_k}{b + ic\omega_k}, -\frac{\sqrt{b}(-a + i\omega_k)}{b + ic\omega_k}\right)e^{is\omega_k\tau_k},$$

where D is a constant such that $\langle q^*(s), q(\theta)\rangle = 1$. By (17), we get

$$\begin{aligned}\langle q^*(s), q(\theta)\rangle &= \bar{D}(1, \bar{\alpha^*}, \bar{\beta^*})(1, \alpha, \beta)^T - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{D}(1, \bar{\alpha^*}, \bar{\beta^*})e^{-i(\xi-\theta)\omega_k\tau_k}d\eta(\theta)(1, \alpha, \beta)^T e^{i\xi\omega_k\tau_k}d\xi \\ &= \bar{D}\{1 + \alpha\bar{\alpha^*} + \beta\bar{\beta^*} + k\tau_k\alpha\bar{\alpha^*}e^{-i\omega_k\tau_k}\}.\end{aligned} \tag{18}$$

Therefore, we can choose D as

$$D = \frac{1}{\{1 + \alpha\bar{\alpha^*} + \beta\bar{\beta^*} + k\tau_k\alpha\bar{\alpha^*}e^{-i\omega_k\tau_k}\}}.$$

Using the same notation as in [14], we will compute the coordinate to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of (15) when $\mu = 0$. Define

$$z(t) = \langle q^*, u_t\rangle, \quad W(t, \theta) = u_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\}. \tag{19}$$

On the center manifold C_0, we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots,$$

where z and $\bar{z}$ are local coordinates for the center manifold C_0 in the directions of q^* and $\bar{q}^*$. Note that W is real if u_t is real, so we deal with real solutions only. For solution $u_t \in C_0$, since $\mu = 0$, we have

$$\begin{aligned}\dot{z}(t) &= i\omega_k\tau_k z + \langle q^*(\theta), f(0, W(z(t), \bar{z}(t), \theta) + 2\mathrm{Re}\{z(t)q(\theta)\})\rangle \\ &= i\omega_k\tau_k z + q^*(0)f(0, W(z(t), \bar{z}(t), 0) + 2\mathrm{Re}\{z(t)q(0)\}).\end{aligned}$$

Let $f(0, W(z(t), \bar{z}(t), 0) + 2\mathrm{Re}\{z(t)q(0)\}) = f_0(z, \bar{z})$, then

$$\dot{z}(t) = i\omega_k\tau_k z + q^*(0)f_0(z, \bar{z}).$$

We rewrite this equation as

$$\dot{z}(t) = i\omega_k\tau_k z + g(z, \bar{z}),$$

where

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{20}$$

Since $q(\theta) = (1, \alpha, \beta)^T e^{i\theta\omega_k\tau_k}$ and $u_t(\theta) = (u_{1t}(\theta), u_{2t}(\theta), u_{3t}(\theta)) = W(t, \theta) + z(t)q(\theta) + \bar{z}(t)\bar{q}(\theta)$, we have

$$\begin{aligned}u_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}\frac{z^2}{2} + W_{11}^{(1)}z\bar{z} + W_{02}^{(1)}\frac{\bar{z}^2}{2} + \cdots, \\ u_{2t}(0) &= \alpha z + \bar{\alpha}\bar{z} + W_{20}^{(2)}\frac{z^2}{2} + W_{11}^{(2)}z\bar{z} + W_{02}^{(2)}\frac{\bar{z}^2}{2} + \cdots, \\ u_{3t}(0) &= \beta z + \bar{\beta}\bar{z} + W_{20}^{(3)}\frac{z^2}{2} + W_{11}^{(3)}z\bar{z} + W_{02}^{(3)}\frac{\bar{z}^2}{2} + \cdots.\end{aligned}$$

From (20), we have

$$\begin{aligned}g(z, \bar{z}) &= q^*(0)f_0(z, \bar{z}) \\ &= \bar{D}\tau_k(1, \alpha^*, \beta^*)\begin{pmatrix}0 \\ -u_{1t}(0)u_{3t}(0) \\ u_{1t}(0)u_{2t}(0)\end{pmatrix} \\ &= \bar{D}\tau_k\Bigg[-\alpha^*\left(z + \bar{z} + W_{20}^{(1)}\frac{z^2}{2} + W_{11}^{(1)}z\bar{z} + W_{02}^{(1)}\frac{\bar{z}^2}{2} + \cdots\right)\left(\beta z + \bar{\beta}\bar{z} + W_{20}^{(3)}\frac{z^2}{2} + W_{11}^{(3)}z\bar{z} + W_{02}^{(3)}\frac{\bar{z}^2}{2} + \cdots\right) \\ &\quad + \beta^*\left(z + \bar{z} + W_{20}^{(1)}\frac{z^2}{2} + W_{11}^{(1)}z\bar{z} + W_{02}^{(1)}\frac{\bar{z}^2}{2} + \cdots\right)\left(\alpha z + \bar{\alpha}\bar{z} + W_{20}^{(2)}\frac{z^2}{2} + W_{11}^{(2)}z\bar{z} + W_{02}^{(2)}\frac{\bar{z}^2}{2} + \cdots\right)\Bigg].\end{aligned}$$

Comparing the coefficients with (20), we have

$$\begin{aligned} g_{20} &= 2\bar{D}\tau_k(\alpha\bar{\beta^*} - \beta\bar{\alpha^*}),\\ g_{11} &= 2\bar{D}\tau_k(\{\bar{\beta^*}\mathrm{Re}\{\alpha\} - \bar{\alpha^*}\mathrm{Re}\{\beta\}\}),\\ g_{02} &= 2\bar{D}\tau_k(\bar{\beta^*}\bar{\alpha} - \bar{\alpha^*}\bar{\beta}),\\ g_{21} &= -\bar{D}\tau_k\bar{\alpha^*}[2W_{11}^{(3)}(0) + W_{20}^{(3)}(0) + 2\beta W_{11}^{(1)}(0) + \bar{\beta}W_{20}^{(1)}(0)]\\ &\quad + \bar{D}\tau_k\bar{\beta^*}[2W_{11}^{(2)}(0) + W_{20}^{(2)}(0) + 2\alpha W_{11}^{(1)}(0) + \bar{\alpha}W_{20}^{(1)}(0)]. \end{aligned} \tag{21}$$

We still need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. From (16) and (19), we have

$$\dot{W} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = \begin{cases} A(0)W - 2\mathrm{Re}\{\bar{q^*}(0)f_0q(\theta)\}, & \theta \in [-1, 0)\\ A(0)W - 2\mathrm{Re}\{\bar{q^*}(0)f_0q(\theta)\} + f_0, & \theta = 0. \end{cases} \tag{22}$$

Let

$$H(z, \bar{z}, \theta) = \begin{cases} 2\mathrm{Re}\{\bar{q^*}(0)f_0q(\theta)\}, & \theta \in [-1, 0)\\ 2\mathrm{Re}\{\bar{q^*}(0)f_0q(\theta)\} + f_0, & \theta = 0. \end{cases}$$

We can rewrite (22) as

$$\dot{W} = A(0)W + H(z, \bar{z}, \theta),$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{23}$$

From (22) and (23) and the definition of W, expanding the series and comparing the coefficients, we have

$$(A(0) - 2i\omega_k\tau_k)W_{20}(\theta) = -H_{20}(\theta), A(0)W_{11}(\theta) = -H_{11}(\theta), \ldots. \tag{24}$$

From (22), we know that for $\theta \in [-1, 0)$,

$$H(z, \bar{z}, \theta) = -\bar{q^*}(0)f_0q(\theta) - q^*(0)\bar{f_0}\bar{q}(\theta) = -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta).$$

Comparing the coefficients with (23), we obtain

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \qquad H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{25}$$

From (24) and (25) and the definition of $A(0)$,

$$\dot{W_{20}} = 2i\omega_k\tau_kW_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta).$$

Substitute $q(\theta) = (1, \alpha, \beta)^T e^{i\theta\omega_k\tau_k}$ into the last equation, we can obtain the solution of it, which reads

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_k\tau_k}q(0)e^{i\theta\omega_k\tau_k} + \frac{i\bar{g}_{02}}{3\omega_k\tau_k}\bar{q}(0)e^{-i\theta\omega_k\tau_k} + E_1e^{2i\theta\omega_k\tau_k}, \tag{26}$$

and similarly

$$W_{11}(\theta) = \frac{ig_{11}}{\omega_k\tau_k}q(0)e^{i\theta\omega_k\tau_k} + \frac{i\bar{g}_{11}}{\omega_k\tau_k}\bar{q}(0)e^{-i\theta\omega_k\tau_k} + E_2, \tag{27}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T \in R^3$ and $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T \in R^3$ are constant vectors corresponding to the initial conditions of the differential equations respectively.

Finally, we will seek the values of E_1 and E_2. For (24), we have

$$\dot{W}_{20}(\theta) = \int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\theta\omega_k\tau_kW_{20}(0) - H_{20}(0), \tag{28}$$

and

$$\dot{W}_{11}(\theta) = \int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{29}$$

where $\eta(\theta) = \eta(\theta, 0)$. From Eq. (22), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau_k(0 - \beta\alpha)^T, \tag{30}$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + 2\tau_k(0 - \mathrm{Re}\{\beta\}\ \mathrm{Re}\{\alpha\})^T. \tag{31}$$

For $i\omega_k\tau_k$ is the eigenvalue of $A(0)$ and $q(0)$ is the corresponding eigenvector, we obtain

$$\left(i\omega_k\tau_k - \int_{-1}^{0} e^{i\theta\omega_k\tau_k} d\eta(\theta)\right) q(0) = 0, \qquad \left(-i\omega_k\tau_k - \int_{-1}^{0} e^{-i\theta\omega_k\tau_k} d\eta(\theta)\right) \bar{q}(0) = 0. \tag{32}$$

Substituting Eqs. (26) and (30) into Eq. (28), we obtain

$$\left(2i\omega_k\tau_k I - \int_{-1}^{0} e^{2i\theta\omega_k\tau_k} d\eta(\theta)\right) E_1 = 2\tau_k(0 - \beta\alpha)^T. \tag{33}$$

That is

$$\begin{pmatrix} 2i\omega_k + a & -a & 0 \\ -c & 2i\omega_k + k + c - ke^{-i\omega_k\tau_k} & \sqrt{b} \\ -\sqrt{b} & -\sqrt{b} & 2i\omega_k \end{pmatrix} E_1 = 2\begin{pmatrix} 0 \\ -\beta \\ \alpha \end{pmatrix}.$$

It follow that

$$E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1}, \qquad E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1}, \qquad E_1^{(3)} = \frac{\Delta_{13}}{\Delta_1},$$

where

$$\Delta_{11} = 2\begin{vmatrix} 0 & -a & 0 \\ -\beta & 2i\omega_k + k + c - ke^{-i\omega_k\tau_k} & \sqrt{b} \\ \alpha & -\sqrt{b} & 2i\omega_k \end{vmatrix}, \qquad \Delta_{12} = 2\begin{vmatrix} 2i\omega_k + a & 0 & 0 \\ -c & -\beta & \sqrt{b} \\ -\sqrt{b} & \alpha & 2i\omega_k \end{vmatrix},$$

$$\Delta_{13} = 2\begin{vmatrix} 2i\omega_k + a & -a & 0 \\ -c & 2i\omega_k + k + c - ke^{-i\omega_k\tau_k} & -\beta \\ -\sqrt{b} & -\sqrt{b} & \alpha \end{vmatrix}, \qquad \Delta_1 = \begin{vmatrix} 2i\omega_k + a & -a & 0 \\ -c & 2i\omega_k + k + c - ke^{-i\omega_k\tau_k} & \sqrt{b} \\ -\sqrt{b} & -\sqrt{b} & 2i\omega_k \end{vmatrix}.$$

Similarly, substituting Eqs. (27) and (31) into Eq. (29), we have

$$\begin{pmatrix} a & -a & 0 \\ -c & c & \sqrt{b} \\ -\sqrt{b} & -\sqrt{b} & 0 \end{pmatrix} E_2 = 2\begin{pmatrix} 0 \\ -\mathrm{Re}\{\beta\} \\ \mathrm{Re}\{\alpha\} \end{pmatrix}.$$

It follows that

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2}, \qquad E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2}, \qquad E_2^{(3)} = \frac{\Delta_{23}}{\Delta_2},$$

where

$$\Delta_{21} = 2\begin{vmatrix} 0 & -a & 0 \\ -\mathrm{Re}\{\beta\} & c & \sqrt{b} \\ \mathrm{Re}\{\alpha\} & -\sqrt{b} & 0 \end{vmatrix}, \qquad \Delta_{22} = 2\begin{vmatrix} a & 0 & 0 \\ -c & -\mathrm{Re}\{\beta\} & \sqrt{b} \\ -\sqrt{b} & \mathrm{Re}\{\alpha\} & 0 \end{vmatrix},$$

$$\Delta_{23} = 2\begin{vmatrix} a & -a & 0 \\ -c & c & -\mathrm{Re}\{\beta\} \\ -\sqrt{b} & -\sqrt{b} & \mathrm{Re}\{\alpha\} \end{vmatrix}, \qquad \Delta_2 = \begin{vmatrix} a & -a & 0 \\ -c & c & \sqrt{b} \\ -\sqrt{b} & -\sqrt{b} & 0 \end{vmatrix}.$$

Consequently, we can determine $W_{20}(0)$ and $W_{11}(0)$, thus, all g_{ij} can be determined by (21).

Following the basic idea of [32] and the method in [33], one can draw the conclusion about the bifurcation direction and the stability of the Hopf bifurcation, which are determined by the following parameters:

$$\begin{aligned} C_1(0) &= \frac{i}{2\omega_k\tau_k}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\left\{\frac{d\lambda(\tau_k)}{d\tau}\right\}}, \\ T_2 &= -\frac{\mathrm{Im}\, C_1(0) + \mu_2 \mathrm{Im}\left\{\frac{d\lambda(\tau_k)}{d\tau}\right\}}{\omega_k\tau_k}, \\ \beta_2 &= 2\mathrm{Re}\{C_1(0)\}. \end{aligned} \tag{34}$$

Therefore, we have the main results in this section.

Theorem 3.1. *In (34), μ_2 determines the direction of the Hopf bifurcation, if $\mu_2 > 0(\mu_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_k(\tau < \tau_k)$; β_2 determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are orbitally stable (unstable) if $\beta_2 < 0(\beta_2 > 0)$, and T_2 determines the period of the bifurcating periodic solutions: the period increase (decreases) if $T_2 > 0(T_2 < 0)$.*

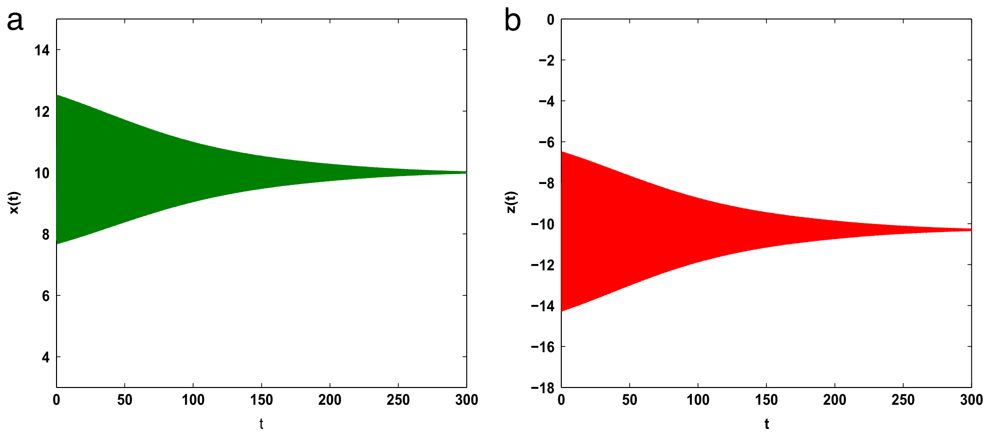

Fig. 2. The equilibrium E_1 is asymptotically stable for system (2) with parameter values $(a, b, c, k) = (10, 100, 10.3, 2)$ and initial values $(9, 8, -8)$ when $\tau = 0.01$.

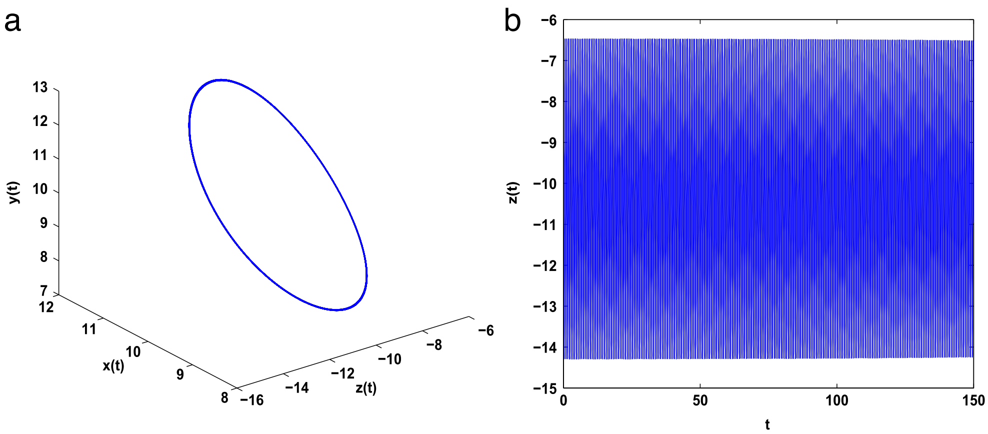

Fig. 3. Bifurcating periodic solution for system (2) with parameter values $(a, b, c, k) = (10, 100, 10.3, 2)$ and initial values $(9, 8, -8)$ when $\tau = 0.0146 < \tau_2^0$ is close to τ_2^0: (a) Phase portrait; (b) Time series of $z(t)$.

4. Numerical results

In the previous section, we have proofed the existence of Hopf bifurcation of system (2). In this section, we will confirm our theoretical analysis by numerical simulation. We choose a set of parameter values and consider the following system:

$$\begin{cases} \dot{x} = 10(y - x) \\ \dot{y} = -10.3y - xz + k[y(t-\tau) - y] \\ \dot{z} = -100 + xy, \end{cases} \tag{35}$$

which has two equilibria $E_1(10, 10, -10.3)$, $E_2(-10, -10, -10.3)$. When $\tau = 0$, the equilibria $E_{1,2}$ are asymptotically stable, system (35) has a chaotic attractor, as shown in Fig. 1. From the discussion in Section 2, if we choose $k = 2$, we obtain that Eq. (10) has two positive roots $\omega_1 = 9.68878$ and $\omega_2 = 9.80433$. Therefore, there are, respectively,

$$\tau_1^j = 0.042965 + 0.648501j, \qquad \tau_2^j = 0.022979 + 0.640858j,$$

where $j = 0, 1, 2, \ldots$. From the formula (34), it follows that $C_1(0) = 0.00139696 - 0.00193273i$, $\mu_2 = -0.00247794$, $T_2 = 0.0733417$, $\beta_2 = 0.00279392$. Thus, the equilibria $E_{1,2}$ is stable for $0 < \tau < \tau_2^0$ as is illustrated by the computer simulations (see Fig. 2(a)–(b)). When τ passes through the critical value τ_2^0, E_1 loses its stability and a Hopf bifurcation occurs. Since $\mu_2 < 0$ and $\beta_2 > 0$, the Hopf bifurcation is subcritical and the direction of the Hopf bifurcation is $\tau < \tau_2^0$ and these bifurcating periodic solutions from E_1 are unstable, which are depicted in Fig. 3(a)–(b). On the other hand, the numerical simulations show that the bifurcating periodic solutions disappear when the delay $\tau > \tau_2^0$, and chaos occurs again. This is shown in Fig. 4(a)–(b).

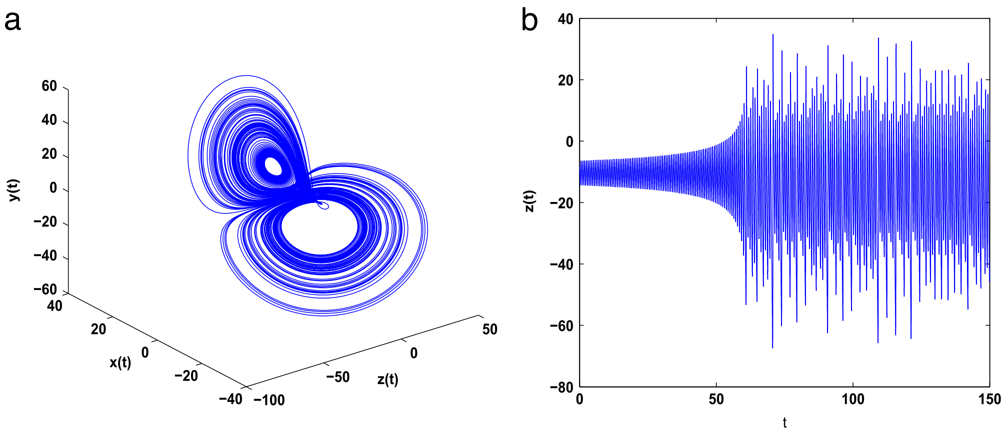

Fig. 4. The equilibria $E_{1,2}$ are both unstable and chaos occurs again for system (2) and initial values $(9, 8, -8)$ when $\tau = 0.03 > \tau_2^0$ is closer to τ_2^0: (a) Chaotic attractor of system (2); (b) Time series of $z(t)$.

5. Conclusion

At least to our knowledge, very little research has been conducted on bifurcation in a chaotic system coexisting with two stable node-foci with time delay feedback. In this paper, we have developed a control model for the system proposed by Yang et al. through time delay feedback control laws, and obtained the conditions that Hopf bifurcation occurs and the stability of equilibria. We also investigate the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions, by using the center manifold theory and normal form method. Our theoretical results and numerical simulations show that the chaos phenomena of system (1) can be controlled by delay. As the delay increases further, the numerical simulations show that the periodic solution disappears and the chaos attractor appears again. The obtained results can also be applied to the control and anti-control of chaos phenomena of system (1) only with two stable node-foci. There are still abundant and complex dynamical behaviors and the topological structure of the new system should be completely and thoroughly investigated and exploited. It is expected that more detailed theory analysis and simulation investigations about this letter will be provided in a forthcoming study.

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